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A Totally Volume-Integrated Discontinuous Galerkin Method with Strong Stability Preserving Time Integration for Periodic Problems on the Interval [-π, π]

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Abstract

This work presents solutions of hyperbolic partial differential equations on periodic domains using a high-order numerical scheme. A Totally Volume-Integrated Discontinuous Galerkin (TVI-DG) method is developed explicitly for spatial discretization of the periodic interval [-π, π], utilizing an orthogonal polynomial basis inspired by Fourier. The integration of semi-discrete systems in time using Second and third-order Strong Stability Preserving (SSP) schemes, also defined on the same periodic interval, which are designed to have better stability properties compared to classical SSP methods. The spatial and temporal discretizations are derived from a unified weighted residual formulation using a standard Galerkin approach. The efficiency of the proposed method is proved through the linear advection equation and the one-dimensional heat equation. The numerical results of the conducted benchmark test cases confirm the accuracy and efficiency of the proposed methods. A comparison between the shape function of this work [-π, π] and those for the standard interval [0, 1] shows that the numerical results are equivalent at a time equal to 2π, highlighting the impact of the computational domain on solution behavior. This work is part of a series extending high-order DG methods to general intervals using generalized polynomial bases.

Keywords: Polynomial Discretization, Hyperbolic Conservation Laws, TVI-DG Method, SSP Time Schemes, Orthogonal Polynomials, Periodic Intervals.

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طريقة جالركين الغير متصلة متكاملة كليًا مع الحجم، مع تكامل زمني قوي يحافظ على الاستقرار، لحل مسائل دورية على الفترة [−π, π]

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الملخص

يقدم هذا البحث حلولًا لمعادلات تفاضلية جزئية زائدية على نطاقات دورية باستخدام مخطط عددي عالي الرتبة. تم تطوير طريقة جالركين غير متصلة متكاملة كليًا مع الحجم (TVI-DG) خصيصًا للتقسيم المكاني للفترة الدورية [−π، π]، باستخدام أساس متعدد الحدود المتعامد المستوحى من فوريه. تكامل الأنظمة شبه المنفصلة في الزمن باستخدام مخططات الحفاظ على الاستقرار القوي (SSP) من الرتبتين الثانية والثالثة، والمُعرّفة أيضًا على نفس الفترة الدورية، والتي صُممت لتكون ذات خصائص استقرار أفضل مقارنةً بأساليب SSP التقليدية. يتم اشتقاق التقديرات المكانية والزمانية من صيغة موحدة للبقايا الموزونة باستخدام نهج جاليركين القياسي. تم إثبات كفاءة الطريقة المقترحة من خلال معادلة الحمل الخطي ومعادلة الحرارة أحادية البعد. تؤكد النتائج العددية لحالات الاختبار المعيارية التي تم إجراؤها على دقة وكفاءة الطرق المقترحة. أظهرت المقارنة بين دالة الشكل في هذا العمل [−π, π] وتلك الخاصة بالفترة القياسية [0, 1] أن النتائج العددية متكافئة عند زمن يساوي 2π، مما يُبرز تأثير المجال الحسابي على سلوك الحل. يُعد هذا العمل جزءًا من سلسلة دراسات تُوسّع نطاق طرق التوليد التفاضلي عالي الرتبة لتشمل فترات عامة باستخدام قواعد متعددة الحدود معممة.

الكلمات المفتاحية :التقطيع متعدد الحدود، قوانين الحفظ الزائدية، طريقة TVI-DG، مخططات زمنية SSP، متعددات الحدود المتعامدة، فترات دورية.

1. Introduction

This is the third article of a series [1, 2] devoted to the construction and study of the so-called Totally Volume-Integrated Discontinuous Galerkin (TVI-DG) method. The numerical solution of hyperbolic conservation laws, characterized by their potential to develop discontinuities (shocks) even from smooth initial data, has been a

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central challenge in computational physics for decades. The Discontinuous Galerkin (DG) method has emerged as a powerful framework for addressing this challenge, combining the high-order accuracy of spectral methods with the geometric flexibility and inherent conservativity of finite volume schemes. The evolution of DG methods can be broadly categorized into two paradigms: the **nodal DG approach**, which interpolates solution values at specific points within an element [3], and the **modal DG approach**, which represents the solution as a linear combination of orthogonal basis functions [4].

The standard DG methodology for a conservation law like

$$u_t + \mathbf{F}(u)_x = 0 \tag{1}$$

where u_t represents the conservative variable, and $\mathbf{F}(u)$ stands for the conservative flux, which involves a two-step process. First, the spatial derivatives are discretized, transforming the Partial Differential Equation (PDE) into a system of Ordinary Differential Equations (ODEs) often referred to as a **semi-discrete system** [5, 6], which can be expressed as:

$$u_t = -Lu. \tag{2}$$

Here, L is a discrete operator representing the spatial discretization. The second step involves applying a high-order time integration scheme to solve this system. A critical requirement for this time stepper is the **Strong Stability Preserving (SSP)** property, which ensures nonlinear stability under a CFL condition by preserving the stability properties of the forward Euler method [7, 8].

A key design choice in modal DG methods is the selection of the orthogonal polynomial basis, which is traditionally tied to the computational domain. Legendre and Chebyshev polynomials, special cases of the Jacobi polynomials, have become the de facto standard for canonical intervals like [-1, 1] and [0, 1] due to their superior approximation properties [9, 10]. The mathematical foundation for this approach is the representation of the solution $u(x)$ as a linear combination of an orthogonal system of functions [11, 12]:

$$u(x) = \sum_0^\infty c_n \phi_n \tag{3}$$

This set of real-valued functions $\phi_n(x)$ is said to be orthogonal with respect to a weight function $w(x)$ on the interval [a, b] if the set satisfies the following condition:

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$$\int_a^b w(x)\,\phi_m(x)\phi_n(x)dx = 1 \quad \text{if } m = n \text{ and 0 otherwise} \qquad (4)$$

The expansion coefficients c_n for the orthogonal expansion can be computed by the projection:

$$c_n = \frac{\int_a^b w(x)u(x)\phi_n(x)}{\|\phi_n\|^2} \qquad (5)$$

where $w(x)$ is the polynomial weight functions and $\|\phi_n\|^2$ is the squared norm of the basis function ϕ_n. Note that we use $w(x)$ and $w(t)$ to denote polynomial weighting functions, distinguishing them from the governing equation's weighting functions $W(x)$ and $W(t)$. For problems defined on unbounded or semi-infinite domains, Laguerre and Hermite polynomials offer a natural basis [13]. However, for problems with inherent periodicity, the Fourier basis on an interval like [-π, π] is the optimal choice. While Kopriva [12] and others [14] have extensively used this framework for spectral methods, its integration into a modern DG formalism, **particularly one that applies the same orthogonal projection principle *consistently to both space and time* discretizations on a non-standard interval,** remains underexplored. Furthermore, the associated time integration schemes are almost exclusively designed for the standard [0, 1] interval, potentially introducing inefficiencies.

This work is part of a series [1, 2] that aims to bridge this gap by constructing a high-order numerical framework where both the spatial and temporal discretizations are derived self-consistently for a specific interval using the projection method defined by Equations (3)-(5). In this paper, we focus on the periodic interval [-π, π]. Our primary contributions are:

1. The formulation of a Totally Volume-Integrated Discontinuous Galerkin (TVI-DG) method for spatial discretization on [-π, π] using this orthogonal projection principle.
2. The derivation of novel second and third-order Strong Stability Preserving (SSP) time integration schemes explicitly designed for the [-π, π] time interval, with optimized coefficients that distinguish them from classical SSP-RK methods.
3. A comprehensive numerical validation of the proposed space-time framework on the linear advection and heat equations, demonstrating high-order accuracy and revealing

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new insights into the influence of the computational domain on solution behavior.

The paper is organized as follows: Section 2 details the TVI-DG spatial discretization for the [-π, π] interval. Section 3 introduces the new SSP time discretization schemes. Section 4 presents numerical results, and Section 5 provides concluding remarks.

2. TVI-DG Spatial discretization in Interval [-π, π]

For clarity in subsequent discussions, this section provides a concise overview of the discontinuous Galerkin (DG) semi-discretization method for solving partial differential equations (PDEs). We begin with the one-dimensional conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \qquad (6)$$

The numerical solution of Equation (6) is computed over the computational domain Ω, subjected to proper initial and boundary conditions. The weighted residual form is derived by multiplying Equation (6) by a test function $W(x)$ and integrating by parts over the domain Ω:

$$\int_{\Omega}[W(x)u_t - W_x F(u)]\, d\Omega + \int_{\Gamma}[F(u).W(x)]\,.\mathbf{n}d\Gamma = 0 \qquad (7)$$

where **n** represents the unit normal vector to the domain boundary Γ that subdivides the computational domain Ω into N non-overlapping elements:

$$\Omega = \sum_1^N \Omega_h \qquad (8)$$

yielding the element-wise formulation:

$$\int_{\Omega_h}[W_h(x)u_t - W_x F(u_h)]\, d\Omega_h + \int_{\Gamma_h}[F_b(u_h).W_h(x)]\,.\mathbf{n}d\Gamma_h = 0 \qquad (9)$$

where $F(u_h)$ and $W_h(x)$ are the flux and test function over the element domain, and $F_b(u_h)$ is the flux over the element boundaries. Omitting the element subscript (h) for simplicity and applying the divergence theory to the boundary term (last term) in Equation (9) yields:

$$\int_{\Omega}\left[Wu_t - \frac{\partial W}{\partial x}F(u) + (\frac{\partial W}{\partial x} + W\frac{\partial}{\partial x})F_b\right]d\Omega = 0 \qquad (10)$$

This defines the totally volume discontinuous Galerkin method (TVI-DG) [15].

In this paper the element domain (interval) Ω = [- π, π] or (a = -π and b = π), $w(x)$ =1 and with $u(x) = W(x)$ (the standard Galerkin

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method, otherwise known as the Petrov-Galerkin method). For spatial discretization for the second-order: basis $\{\phi_0 = 1, \phi_1 = x\}$, and for the third-order: additional basis $\phi_2 = x^2$, substituting these into equations (3-5) completes the spatial discretization.

3. Strong Stability Preserving (SSP) - Time Discretization for Interval [-π, π]

Following spatial discretization, we consider the semi-discrete form of Equation (10), isolating the spatial terms on the right-hand side. Then multiplying the equation by a temporal weighting function $W(t)$ and integrating over the time domain $\Omega = [-\pi, \pi]$ yields:

$$\int_{\Omega} W(t) Q_t = W(t) Lu \qquad (11)$$

where $Q_t = W(x)\frac{\partial u}{\partial t}$ represents the weighted temporal derivative, and Lu is the spatial discretized operator of u function, which is constant in time.

For this paper, we define the temporal basis function $u(t)$ and weight functions $W(t)$ over the domain $\Omega = [-\pi\ \pi]$ with $\{\phi_0 = 1, \phi_1 = t\}$ for the second-order polynomial shape, and add $\{\phi_2 = t^2\}$ for the third-order shape function, and then substituting these into equations (3-5) and simplifying, the second and third-order SSP-time discretization are given as follows:

The second-order SSP time discretization scheme for the [-π π] interval is given as:

let $u^n = Q^n$. The update steps are:

1. Initialization:

$$u^0 = u^n$$

2. First stage:

$$u^1 = u^0 + 6.28318530700648292253\ Lu^0\ \Delta t$$

3. Second stage:

$$u^2 = u^0 + 3.141592653412999425\ Lu^0\ \Delta t + 3.141592653412999425\ Lu^1\ \Delta t \qquad (12)$$

where Lu^n is the spatial discretization at step n and Δt is the time step, $n = 0, 1, 2$

The third-order SSP scheme is given as:

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1. Initialization:

$$u^0 = u^n$$

2. Predictor stage:

$$u^2 = u^0 + 6.283185307006482922 53\ Lu^0\ \Delta t$$

where u^2 as a guess value.

3. First stage:

$$u^1 = 0.75\ u^0 + 0.25\ u^2 + 1.570796328436\ Lu^0$$

4. Second stage:

$$u^2 = u^0 + 6.283185309105\ Lu^1\ \Delta t$$

5. Third stage:

$$u^3 = u^0 + 1.047197552291\ Lu^0 + 4.188790206070\ Lu^1 + 1.047197552291\ Lu^2 \quad (13)$$

For comparison with the standard SSP-time method, for reference, the classical second-order SSP-RK method is given as:

1. First stage:

$$u^1 = u^0 + Lu^0\ \Delta t$$

2. Second stage:

$$u^2 = u^0 + (0.5Lu^0 + 0.5Lu^1)\Delta t \quad (14)$$

4. Numerical Results

All computations were conducted on an HP laptop equipped with an Intel® Core™ i7-10510U processor running at 1.80 GHz (up to 2.30 GHz) and 8 GB RAM. The operating system used was Ubuntu 24.04 LTS with the Linux distribution. The codebase was implemented in C++ and compiled using g++. Within this study, the Total Volume-Discontinuous Galerkin method was implemented for all test cases. The global error was calculated as the difference between the exact and numerical solutions. The discrete L_1 norm error is given as:

$$L_1 = \frac{\sum_{i=1}^{N}(\sum_{j=1}^{edof}|u_{\text{exact},j} - u_{i,j}|)}{tdof} \quad (15)$$

where N is the total number of elements, $edof$ is the element degree of freedom (local DOF), and $tdof$ is the total degree of freedom ($N \times edof$).

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In all the series of papers addressing the spatial and temporal discretization of polynomials across intervals beyond [0, 1], identical sets of test cases were utilized.

4.1 The linear advection equation test case

The first benchmark problem solves the linear advection equation, considered in many references, such as [16, 17].

$$\left(\frac{\partial u}{\partial t}\right) + \frac{\partial F(u)}{\partial x} = 0, \quad \mathrm{F}(u) = u \tag{16}$$

The initial condition is given as $u(x, 0) = \sin(x)$ with periodic boundary conditions.

The problem domain [-π, π] is divided into N equally spaced elements. Approximation solutions are derived from 2nd and 3rd order polynomials within the interval [-π, π]. Furthermore, SSP time discretizations of 2nd and 3rd orders, given in equations (12) and (13), are used for time evaluation. The exact solution is represented as $u(x, t) = \sin(x - t)$. Figure 1 illustrates the numerical solutions within the interval [-π, π] with N = 100 elements at T = 1, using a 2nd order polynomial for both space and time shape functions. The accuracy orders and L_1 errors using N = 160 elements are shown in Figure 2 and given in Tables 1 and 2 for polynomial orders k = 2 and 3, respectively.

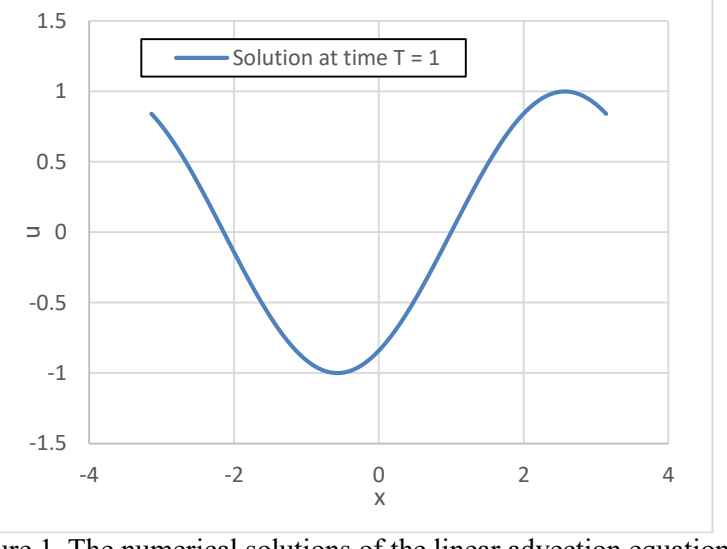

Figure 1. The numerical solutions of the linear advection equation at time T = 1 for the interval [-π, π], using N = 100 elements.

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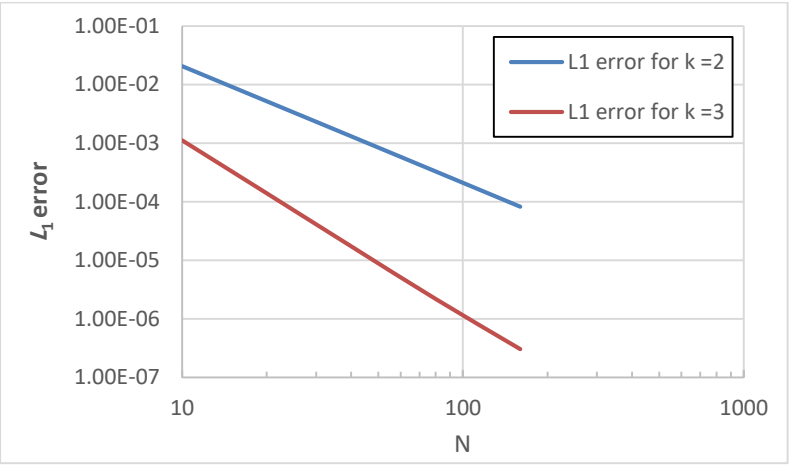

Figure 2. L_1 error of polynomials of orders $k2$ and $k3$ for the 1D linear advection equation for the interval $[-\pi, \pi]$.

Table 1. The order of accuracy and L1 error of the linear advection equation at time T = 1 for a polynomial of order k = 2 $[-\pi, \pi]$

N	L_1 error	L_1 Order of accuracy
10	2.06116050216110e-02	-
20	5.18448524943224e-03	1.991184
40	1.30569857098123e-03	1.989379
80	3.27130411461764e-04	1.996884
160	8.18019352136450e-05	1.999659

Table 2. The order of accuracy and L_1 error of the linear advection equation at time $T = 1$ for a polynomial of order $k = 3$ $[-\pi, \pi]$

N	L_1 error	L_1 Order of accuracy
10	1.11302727661263e-03	-
20	1.39357099935344e-04	2.997631
40	1.73082313575405e-05	3.009256
80	2.18519093073298e-06	2.985627
160	3.04909076574088e-07	2.841308

4.2 The one-dimensional heat equation test case

The second benchmark test is the one-dimensional heat equation, also known as the 1D-diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{17}$$

The equation is subjected to the following initial and boundary conditions:

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$$u(x,0) = \sin(x),\ u(-\pi,t) = u(\pi,t) = 0$$

To start the numerical solution, the second-order partial differential equation is reduced to two first-order differential equations (local discontinuous Galerkin method). The equations were solved in the domain [-π, π], which is divided into N = 40 equally spaced elements.

Let $Q = \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t} = \frac{\partial Q}{\partial x}$. These two equations can then be solved numerically.

Figure 3 illustrates the comparison between the numerical solutions obtained from shape functions represented by polynomials at the intervals [-π, π] and [0, 1] of order k = 2, at time T = 0.5 and 1. The illustration reveals that as time increases, the peak points decrease. On the other hand, the numerical results and Figure 3 show that the peak values are 0.605531 and 0.367076 for the standard polynomial interval [0, 1], respectively. In the interval [-π, π], the peak values at time T = 0.5 and 1 are 0.922938 and 0.852047, respectively. Finally, at time T = 2π, the maximum value is 0.369869, which is equivalent to the value obtained by the standard polynomial at time T = 1.

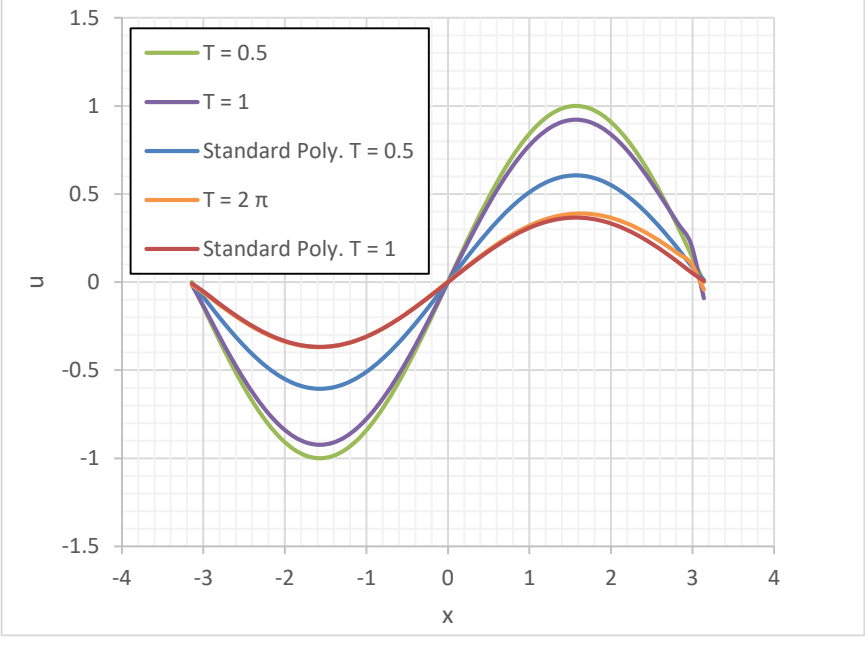

Figure 3. The comparison between numerical solutions of sin(x), 1D heat equation for the intervals [-π, π] and [0, 1].

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5. CONCLUSIONS

In this work, we present a detailed investigation of polynomial discretization schemes of hyperbolic conservation laws through the TVI-DG method and SSP time discretizations. With the application of orthogonal polynomials suitable for specific intervals, we were able to develop second and third-order spatial discretization and SSP-time schemes for the periodic domain $[-\pi, \pi]$. Numerical results show the stability, accuracy, and effectiveness of the developed schemes in solving test problems. The study contributes to the enhancement of knowledge and the use of polynomial-based techniques in numerical solutions of hyperbolic conservation laws, presenting a capable approach for subsequent research and progress in computational fluid dynamics and related fields.

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